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# Bicomplex formulation and Moyal deformation of (2 + 1)-dimensional Fordy–Kulish systems

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**Abstract**
Using bicomplex formalism we construct generalizations of Fordy–Kulish systems of matrix nonlinear Schrödinger equations on two-dimensional space-time in two respects. Firstly, we obtain corresponding equations in three space-time dimensions. Secondly, a Moyal deformation is applied to the space-time coordinates and the ordinary product of functions replaced by the Moyal product in a suitable way. Both generalizations preserve the existence of an infinite set of conservation laws.

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## 1. Introduction

A *bicomplex* is an $\mathbb{N}_0$-graded linear space (over $\mathbb{R}$ or $\mathbb{C}$) $M = \bigoplus_{s\geqslant 0} M^s$ together with two linear maps $\mathrm{d}, \delta : M^s \to M^{s+1}$ satisfying

$$\mathrm{d}^2 = 0 \qquad \delta^2 = 0 \qquad \mathrm{d}\,\delta + \delta\,\mathrm{d} = 0. \tag{1.1}$$

Associated with a bicomplex is the *linear equation*

$$\delta\chi = \lambda\,\mathrm{d}\,\chi \tag{1.2}$$

where $\chi \in M^0$ and $\lambda$ is a parameter [1]. If it admits a (non-trivial) solution as a (formal) power series $\chi = \sum_{r\geqslant 0} \lambda^r \chi^{(r)}$ in $\lambda$, the linear equation leads to

$$\delta\chi^{(0)} = 0 \qquad \delta\chi^{(r)} = \mathrm{d}\chi^{(r-1)} \qquad r = 1, \ldots, \infty. \tag{1.3}$$

As a consequence, $J^{(r+1)} = \mathrm{d}\chi^{(r)}$, $r = 0, \ldots, \infty$, are $\delta$-exact. These elements of $M^1$ may be regarded as generalized conserved currents [1].

In section 2 we start with a trivial bicomplex. A certain 'dressing' (in the sense of [1]) then leads to a bicomplex formulation of the Fordy–Kulish systems [2] of matrix nonlinear Schrödinger (matrix-NLS) equations[3]. More precisely, our approach leads to an extension of

[3] See also [3]. For some other generalizations of the NLS equation see [4], in particular.

the latter systems from two to three space-time coordinates $t, x, y$ from which the matrix-NLS equations are obtained via the reduction $y = x$.[4] Our $(2+1)$-dimensional systems turn out to be matrix generalizations of a system studied in [7, 8] (see also the references given there).

Furthermore, in section 3 deformation quantization [9] is applied to the space-time coordinates. The ordinary commutative product in the algebra $\mathcal{A}$ of smooth functions on $\mathbb{R}^3$ is replaced with the $*$-product, which is defined by

$$f * h = \boldsymbol{m} \circ \mathrm{e}^{\mathrm{i}\mathcal{P}/2}(f \otimes h) \tag{1.4}$$

where $\boldsymbol{m}(f \otimes h) = f\,h$ for all $f, h \in \mathcal{A}$, and $\mathcal{P} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is given by

$$\mathcal{P} = \vartheta_1\,(\partial_t \otimes \partial_x - \partial_x \otimes \partial_t) + \vartheta_2\,(\partial_t \otimes \partial_y - \partial_y \otimes \partial_t) + \vartheta_3\,(\partial_x \otimes \partial_y - \partial_y \otimes \partial_x) \tag{1.5}$$

with real deformation parameters[5] $\vartheta_j$, $j = 1, 2, 3$. Under complex conjugation, we have $\overline{f * h} = \overline{h} * \overline{f}$ for functions $f, h$. The partial derivatives $\partial_t, \partial_x, \partial_y$ are derivations of the $*$-product. Space-time deformation quantization has been applied recently to various integrable models in [10–12], for example.

Section 4 deals with the conservation laws of the extended and deformed Fordy–Kulish systems. Section 5 gives corresponding generalized ferromagnet equations for the latter systems. Section 6 contains some conclusions.

## 2. Bicomplex formulation of extended Fordy–Kulish systems

We choose the bicomplex space as $M = M^0 \otimes \Lambda$ where $M^0 = C^\infty(\mathbb{R}^3, \mathbb{C}^N)$ denotes the set of smooth maps $\phi : \mathbb{R}^3 \to \mathbb{C}^N$ and $\Lambda = \mathbb{C} \oplus \Lambda^1 \oplus \Lambda^2$ is the exterior algebra of a two-dimensional complex vector space with basis $\tau, \xi$ of $\Lambda^1$ (so that $\tau^2 = \xi^2 = \tau\,\xi + \xi\,\tau = 0$). $M$ becomes a bicomplex with the maps d and $\delta$ defined by

$$\mathrm{d}\phi = \phi_t\,\tau + \phi_x\,\xi \tag{2.1}$$

$$\delta\phi = \phi_y\,\tau + (A - a\,I)\,\phi\,\xi \tag{2.2}$$

where an index denotes a partial derivative with respect to one of the coordinates $t, x, y$ on $\mathbb{R}^3$, for example $\phi_t = \partial_t\phi$. $A$ is a constant $N \times N$ matrix, $I$ the identity matrix, and $a \in \mathbb{C}$.[6] By linearity and $\mathrm{d}(\phi\,\tau + \varphi\,\xi) = (\mathrm{d}\phi)\,\tau + (\mathrm{d}\varphi)\,\xi$ (and correspondingly for $\delta$) the maps d and $\delta$ extend to the whole of $M$. Now we apply a 'dressing' to d as follows:

$$\begin{aligned} \mathrm{D}\phi &= \mathrm{d}\phi + \delta(L\,\phi) - L\,\delta\phi \\ &= (\phi_t + L_y\phi)\,\tau + (\phi_x + [A, L]\,\phi)\,\xi \end{aligned} \tag{2.3}$$

with an $N \times N$ matrix $L$ and $[A, L] = A\,L - L\,A$. Besides $\delta^2 = 0$, also $\delta\mathrm{D} + \mathrm{D}\delta = 0$ is identically satisfied. The only nontrivial new bicomplex equation is $\mathrm{D}^2 = 0$, which takes the form

$$L_{yx} - [A, L_t] - [L_y, [A, L]] = 0. \tag{2.4}$$

Let us assume that $A$ and $L$ take values in a representation of the Lie algebra $\mathfrak{g}$ of a simple Lie group $G$. Let $K$ be a subgroup of $G$ with Lie algebra $\mathfrak{k}$, and $\mathfrak{m}$ the vector space complement

[4] A different extension of the Fordy–Kulish systems to $2+1$ dimensions obtained by replacing the spectral parameter in the $(1+1)$-dimensional systems by a new partial derivative appeared in [5]. See also [6] for some related work.

[5] Only one of these parameters is actually independent since $\mathcal{P}$ is antisymmetric and thus degenerate in three dimensions.

[6] More precisely, $\mathrm{d}^2$ vanishes identically, $\delta^2 = 0$ requires $(A - a\,I)_y = 0$, and $\mathrm{d}\,\delta + \delta\,\mathrm{d} = 0$ is satisfied iff $(A - a\,I)_t = 0$. This still allows an $x$-dependence of $A - a\,I$. In the following, we will be interested in the possibility of a reduction of the system to two space-time dimensions by setting $y = x$. Then $A - a\,I$ has to be constant.

of $\mathfrak{k}$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$. We assume that the homogeneous space $G/K$ is reductive and moreover symmetric, i.e.

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m} \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{2.5}$$

For a Hermitian symmetric space with a complex structure $J : \mathfrak{m} \to \mathfrak{m}$, $J^2 = -1$, the following conditions hold (cf [2]). There is an element $A \in \mathfrak{g}$ such that $\mathfrak{k} = \ker \mathrm{ad} A$. For a particular scaling of $A$, we have $J = \mathrm{ad} A$ and there is a subset $\theta^+$ of the positive root system such that $\mathfrak{m} = \mathrm{span}\{e_{\pm\alpha} | \alpha \in \theta^+\}$ and $[A, e_{\pm\alpha}] = \pm \mathrm{i}\, e_{\pm\alpha}$ for $\alpha \in \theta^+$. Here $e_\alpha$ belongs to the Cartan–Weyl basis. Moreover, $[e_\alpha, e_\beta] = 0 = [e_{-\alpha}, e_{-\beta}]$ for all $\alpha, \beta \in \theta^+$.

Now we choose $A$ in (2.2) with the above properties. $A$ is then $\mathfrak{k}$-valued. With the decomposition

$$L = Q + P \tag{2.6}$$

where $Q \in \mathfrak{m}$ and $P \in \mathfrak{k}$, the $\mathfrak{k}$- and $\mathfrak{m}$-part of (2.4) reads, respectively,

$$P_{xy} = [Q_y, [A, Q]] \tag{2.7}$$

$$Q_{xy} = [A, Q]_t + [P_y, [A, Q]]. \tag{2.8}$$

According to the above assumptions, $Q$ has a decomposition

$$Q = Q^+ + Q^- \qquad [A, Q^\pm] = \pm \mathrm{i}\, Q^\pm. \tag{2.9}$$

Now one finds that (2.7) can be integrated with respect to $y$. This yields

$$P_x = -\mathrm{i}\, [Q^+, Q^-] \tag{2.10}$$

(up to addition of a $\mathfrak{k}$-valued matrix, which only depends on $t$ and $x$ and which we disregard in the following). Equation (2.8) now leads to

$$\mathrm{i}\, Q^\pm_t \mp Q^\pm_{xy} + \mathrm{i}\, [P_y, Q^\pm] = 0. \tag{2.11}$$

The system of equations (2.10), (2.11) constitutes an extension of the Fordy–Kulish systems [2] to which it reduces when $y = x$. In the latter case, (2.10) determines $P_x$, which can then be eliminated from (2.11). We are then left with the following equations:

$$\mathrm{i}\, Q^\pm_t \mp Q^\pm_{xx} + [[Q^+, Q^-], Q^\pm] = 0. \tag{2.12}$$

**Example.** Let $G = SU(2)$ and $y = x$. The subalgebra of $su(2)$ spanned by $A = (\mathrm{i}/2)\,\sigma_3$ with the Pauli matrix $\sigma_3$ is clearly annihilated by $\mathrm{ad}A$ and generates a $U(1)$ subgroup. Let

$$Q^+ = \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix} \qquad Q^- = \begin{pmatrix} 0 & 0 \\ -\overline{\psi} & 0 \end{pmatrix} \tag{2.13}$$

where $\psi$ is a complex function with complex conjugate $\overline{\psi}$. Then the two equations (2.12) both reduce to the nonlinear Schrödinger equation

$$\mathrm{i}\, \psi_t = -\psi_{xx} - 2\, |\psi|^2 \psi. \tag{2.14}$$

This example is easily generalized [2]. Let us consider the Hermitian symmetric space $SU(N)/S(U(n) \times U(N-n))$ and choose

$$A = \begin{pmatrix} c_1\, I_n & 0 \\ 0 & c_2\, I_{N-n} \end{pmatrix} \tag{2.15}$$

where $I_n$ is the $n \times n$ unit matrix and $c_1, c_2 \in \mathbb{C}$. With

$$Q^+ = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \qquad Q^- = \begin{pmatrix} 0 & 0 \\ -q^\dagger & 0 \end{pmatrix} \tag{2.16}$$

where $q$ is an $n \times (N-n)$ matrix with Hermitian conjugate $q^\dagger$, we obtain

$$[A, Q^{\pm}] = \pm(c_1 - c_2)\, Q^{\pm}. \tag{2.17}$$

The constants $c_1, c_2$ are thus related by $c_1 - c_2 = \mathrm{i}$. Since $A$ must be traceless, we also have $nc_1 + (N-n)c_2 = 0$. Hence

$$c_1 = \frac{N-n}{N}\,\mathrm{i} \qquad c_2 = -\frac{n}{N}\,\mathrm{i}. \tag{2.18}$$

The matrix $P$ must have the form

$$P = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} \tag{2.19}$$

with an $n \times n$ matrix $p$ and an $(N-n)\times(N-n)$ matrix $r$. From (2.10) we obtain the equations

$$p_x = \mathrm{i}\, q\, q^\dagger \qquad r_x = -\mathrm{i}\, q^\dagger\, q \tag{2.20}$$

which are compatible with the unitarity constraints $p^\dagger = -p$ and $r^\dagger = -r$, and with $\mathrm{tr}(p) + \mathrm{tr}(r) = 0$. (2.11) becomes

$$\mathrm{i}\, q_t - q_{xy} + \mathrm{i}\,(p_y\, q - q\, r_y) = 0 \tag{2.21}$$

and its Hermitian conjugate. The reduction $y = x$ leads to the matrix nonlinear Schrödinger equation [2]

$$\mathrm{i}\, q_t - q_{xx} - 2\, q\, q^\dagger\, q = 0. \tag{2.22}$$

The more general systems determined by (2.20) and (2.21) on three-dimensional space-time will be called 'extended matrix-NLS equations' in the following.

Other examples of Hermitian symmetric spaces lead to further matrix nonlinear Schrödinger equations [2] and extensions in the above sense.

## 3. Space-time deformation quantization of the extended matrix-NLS equations

In this section we apply a deformation quantization to the algebra of (smooth) functions on space-time. The bicomplex $(M, \mathrm{D}, \delta)$ introduced in the previous section then generalizes to the deformed noncommutative algebra with the following definition:

$$\begin{aligned} \mathrm{D}\phi &= \mathrm{d}\phi + \delta(L * \phi) - L * \delta\phi \\ &= (\phi_t + L_y * \phi)\, \tau + (\phi_x + [A, L]_* * \phi)\, \xi \end{aligned} \tag{3.1}$$

with $[A, L]_* = A * L - L * A$. The only nontrivial bicomplex equation is still $\mathrm{D}^2 = 0$, which now takes the form

$$L_{yx} - [A, L_t]_* - [L_y, [A, L]_*]_* = 0. \tag{3.2}$$

Of course, the $*$-commutator does not preserve a Lie algebra structure, in general. As a consequence, a decomposition of the last equation like that worked out for Hermitian symmetric spaces in [2] and the previous section does not work, in general. However, in the case of extended matrix-NLS equations only a certain block structure of the matrices entering the bicomplex maps is important. Let $\mathfrak{m}^{\pm}$ be the set of all $N \times N$ matrices of the form of $Q^{\pm}$ in (2.16). Let $\mathfrak{k}$ be the set of all block diagonal matrices (such as $P$ in (2.19)). Then we have $\mathfrak{k} * \mathfrak{k} \subset \mathfrak{k}$, $\mathfrak{k} * \mathfrak{m}^{\pm} \subset \mathfrak{m}^{\pm}$ and $\mathfrak{m}^{\pm} * \mathfrak{k} \subset \mathfrak{m}^{\pm}$. Moreover, since $A$ given in (2.15) with (2.18) is constant, we still have $[A, P]_* = 0$ and $[A, Q^{\pm}]_* = \pm\mathrm{i}\, Q^{\pm}$. Hence, we can proceed with the decomposition $L = Q^+ + Q^- + P$ as in the previous section. The above deformed bicomplex equation now results in the following system:

$$p_x = \mathrm{i}\, q * q^\dagger \qquad r_x = -\mathrm{i}\, q^\dagger * q \tag{3.3}$$

and

$$\mathrm{i}\, q_t - q_{xy} + \mathrm{i}\,(p_y * q - q * r_y) = 0. \tag{3.4}$$

This is a noncommutative version of the corresponding extended matrix-NLS system. The equations (3.3) are consistent with unitarity constraints on $p$ and $r$, but not with $\mathrm{tr}(p)+\mathrm{tr}(r) = 0$. In contrast to the classical case, these matrices can no longer be taken as Lie algebra valued. They have values in the corresponding enveloping algebra instead.

The reduction $y = x$ of the above system leads to the noncommutative matrix nonlinear Schrödinger equation

$$\mathrm{i}\, q_t - q_{xx} - 2\, q * q^\dagger * q = 0 \tag{3.5}$$

which is the matrix version of the noncommutative nonlinear Schrödinger equation treated in [11].

## 4. Conservation laws for the three-dimensional extensions and deformations of matrix-NLS equations

The linear equation associated with the bicomplex underlying the deformed extended matrix-NLS equations of the previous section is

$$\delta\chi = \lambda\, \mathrm{D}\chi \tag{4.1}$$

with a parameter $\lambda$. $\chi$ is taken to be an $N \times n$ matrix of functions. The linear equation is equivalent to the two equations

$$\chi_y = \lambda\,(\chi_t + L_y * \chi) \tag{4.2}$$

$$(A - a)\,\chi = \lambda\,(\chi_x + [A, L]_* * \chi). \tag{4.3}$$

Let us decompose $\chi$ into an $n \times n$ matrix $\alpha$ and an $(N - n) \times n$ matrix $\beta$,

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{4.4}$$

In order to have a nontrivial solution of $\delta\chi^{(0)} = 0$, we choose $a$ as an eigenvalue of $A$. To be more concrete,

$$a = \frac{N - n}{N}\,\mathrm{i} \qquad A - a\, I = -\mathrm{i} \begin{pmatrix} 0 & 0 \\ 0 & I_{N-n} \end{pmatrix} \qquad \chi^{(0)} = \begin{pmatrix} I_n \\ 0 \end{pmatrix}. \tag{4.5}$$

From (4.3) we obtain

$$\alpha_x + \mathrm{i}\, q * \beta = 0 \qquad \beta = \mathrm{i}\,\lambda\,(\beta_x + \mathrm{i}\, q^\dagger * \alpha). \tag{4.6}$$

Assuming that $q$ has a left $*$-inverse, this implies

$$\beta = \mathrm{i}\, q_*^{-1} * \alpha_x \tag{4.7}$$

$$\alpha_x = \mathrm{i}\,\lambda\,(\alpha_{xx} - q_x * q_*^{-1} * \alpha_x + q * q^\dagger * \alpha). \tag{4.8}$$

Furthermore, (4.2) leads to

$$\alpha_y = \lambda\,(\alpha_t + p_y * \alpha + \mathrm{i}\, q_y * q^{-1} * \alpha_x). \tag{4.9}$$

$\alpha$ has a right $*$-inverse at least as a formal power series in $\lambda$, since at zeroth order it is equal to $I_n$. Hence there are $n \times n$ matrices $\rho$, $\sigma$ and $\zeta$ such that

$$\alpha_x = \mathrm{i}\,\lambda\,\rho * \alpha \qquad \alpha_t = \mathrm{i}\,\lambda\,\sigma * \alpha \qquad \alpha_y = \mathrm{i}\,\lambda\,\zeta * \alpha. \tag{4.10}$$

Then (4.8) and (4.9) lead, respectively, to

$$\rho = q * q^\dagger + \mathrm{i}\,\lambda\,(\rho_x - q_x * q_*^{-1} * \rho) - \lambda^2\,\rho * \rho \tag{4.11}$$

$$\lambda\,\sigma - \zeta = \mathrm{i}\,p_y - \mathrm{i}\,\lambda\,q_y * q_*^{-1} * \rho. \tag{4.12}$$

The integrability conditions $\alpha_{xt} = \alpha_{tx}$ and $\alpha_{xy} = \alpha_{yx}$ together with (4.10) yield

$$\rho_t - \sigma_x + \mathrm{i}\,\lambda\,[\rho, \sigma]_* = 0 \tag{4.13}$$

$$\zeta_x - \rho_y - \mathrm{i}\,\lambda\,[\rho, \zeta]_* = 0. \tag{4.14}$$

Differentiation of (4.12) with respect to $x$, using (4.14), $\mathrm{i}\,p_x = -q * q^\dagger$ (cf (3.3)) and (4.11), leads to

$$\sigma_x = (\mathrm{i}\,(\rho_x - q_x * q_*^{-1} * \rho) - \lambda\,\rho * \rho)_y - \mathrm{i}\,(q_y * q_*^{-1} * \rho)_x + \mathrm{i}\,[\rho, \zeta]_*. \tag{4.15}$$

Inserted in (4.13), this yields

$$\rho_t + \mathrm{i}\,(q_y * q_*^{-1} * \rho)_x - (\mathrm{i}\,(\rho_x - q_x * q_*^{-1} * \rho) - \lambda\,\rho * \rho)_y + \mathrm{i}\,[\rho, \lambda\,\sigma - \zeta]_* = 0. \tag{4.16}$$

In terms of the product

$$f \diamond h = \boldsymbol{m} \circ \frac{\sin(\mathcal{P}/2)}{\mathcal{P}/2}(f \otimes h) \tag{4.17}$$

with $\mathcal{P}$ defined in (1.5), the $*$-commutator of two functions can be written as follows:

$$\begin{aligned} \frac{1}{\mathrm{i}}[f, h]_* &= 2\,\boldsymbol{m} \circ \sin(\mathcal{P}/2)\,(f \otimes h) \\ &= (f \diamond (\vartheta_1\,h_x + \vartheta_2\,h_y))_t + (f \diamond (-\vartheta_1\,h_t + \vartheta_3\,h_y))_x - (f \diamond (\vartheta_2\,h_t + \vartheta_3\,h_x))_y. \end{aligned} \tag{4.18}$$

Taking the trace of (4.16), using the last formula and (4.12), we obtain the conservation law

$$\begin{aligned} 0 = \mathrm{tr}(\rho &- \mathrm{i}\,\rho \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)(p_y - \lambda\,q_y * q_*^{-1} * \rho)])_t \\ &+ \mathrm{tr}(\mathrm{i}\,q_y * q_*^{-1} * \rho + \mathrm{i}\,\rho \diamond [(\vartheta_1\,\partial_t - \vartheta_3\,\partial_y)(p_y - \lambda\,q_y * q_*^{-1} * \rho)])_x \\ &+ \mathrm{tr}(\lambda\,\rho * \rho - \mathrm{i}\,(\rho_x - q_x * q_*^{-1} * \rho) \\ &+ \mathrm{i}\,\rho \diamond [(\vartheta_2\,\partial_t + \vartheta_3\,\partial_x)(p_y - \lambda\,q_y * q_*^{-1} * \rho)])_y. \end{aligned} \tag{4.19}$$

Expanding $\rho$ in a formal power series in $\lambda$, i.e.

$$\rho = \sum_{r=0}^{\infty} \lambda^r\,\rho^{(r)} \tag{4.20}$$

(4.11) leads to

$$\rho^{(0)} = q * q^\dagger \qquad \rho^{(1)} = \mathrm{i}\,q * q_x^\dagger \qquad \rho^{(2)} = -q * q^\dagger * q * q^\dagger - q * q_{xx}^\dagger \tag{4.21}$$

and

$$\rho^{(r)} = \mathrm{i}\,(\rho_x^{(r-1)} - q_x * q_*^{-1} * \rho^{(r-1)}) - \sum_{s=0}^{r-2} \binom{r-2}{s} \rho^{(s)} * \rho^{(r-2-s)} \tag{4.22}$$

for $r \geqslant 2$. Inserting this in the expression

$$w = \sum_{r=0}^{\infty} \lambda^r\,w^{(r)} = \mathrm{tr}(\rho - \mathrm{i}\,\rho \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)(p_y - \lambda\,q_y * q_*^{-1} * \rho)]) \tag{4.23}$$

which appears in the above conservation law, an infinite set of conserved densities is obtained, starting with[7]

$$w^{(0)} = \mathrm{tr}(q * q^\dagger - \mathrm{i}\,(q * q^\dagger) \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)\,p_y]) \tag{4.24}$$

$$\begin{aligned} w^{(1)} = \mathrm{tr}(&\mathrm{i}\,q * q_x^\dagger + (q * q_x^\dagger) \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)\,p_y] \\ &+\mathrm{i}\,(q * q^\dagger) \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)(q_y * q^\dagger)]) \end{aligned} \tag{4.25}$$

$$\begin{aligned} w^{(2)} = \mathrm{tr}(&-q * q^\dagger * q * q^\dagger - q * q_{xx}^\dagger + \mathrm{i}\,(q * q^\dagger * q * q^\dagger + q * q_{xx}^\dagger) \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)\,p_y] \\ &-(q * q_x^\dagger) \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)(q_y * q^\dagger)] - (q * q^\dagger) \diamond [(\vartheta_1\,\partial_x + \vartheta_2\,\partial_y)(q_y * q_x^\dagger)]) \end{aligned} \tag{4.26}$$

which in turn can be expanded in (formal) power series in the deformation parameters. For vanishing deformation parameters, the conserved densities $w^{(r)}$ are polynomials in $q, q^\dagger$ and their $x$-derivatives, but no $y$-derivatives. This means that the conserved densities of an extended matrix-NLS system are the same (as polynomials in the fields and their partial derivatives) as those of the corresponding matrix-NLS system (which is obtained from the former by setting $y = x$). This is no longer so after deformation.

## 5. Generalized ferromagnet equations associated with deformed extended Fordy–Kulish systems

A gauge transformation of the (deformed) bicomplex considered in section 3 is a map $g : \mathbb{R}^3 \to G$ such that

$$\mathrm{D}\phi \mapsto \mathrm{D}'\phi' = g_*^{-1}\mathrm{D}(g * \phi) \qquad \delta\phi \mapsto \delta'\phi' = g_*^{-1}\delta(g * \phi) \tag{5.1}$$

for all $\phi \in M$ with $\phi' = g * \phi$. Such a map leaves the bicomplex equations invariant. Let us choose $g$ such that $\mathrm{D}' = \mathrm{d}$. Then

$$g_*^{-1} * g_t = -g_*^{-1} * L_y * g \qquad g_*^{-1} * g_x = -g_*^{-1} * [A, L]_* * g \tag{5.2}$$

and, writing $\mathcal{D}$ instead of $\delta'$, we obtain

$$\mathcal{D}\phi = (\phi_y + R * \phi)\,\tau + (S - a\,I) * \phi\,\xi \tag{5.3}$$

where we have introduced the abbreviations

$$R = g_*^{-1} * g_y \qquad S = g_*^{-1} * A * g. \tag{5.4}$$

Now $\mathcal{D}^2 = 0$ reads

$$S_y = [S, R]_* \tag{5.5}$$

and $\mathrm{d}\mathcal{D} + \mathcal{D}\mathrm{d} = 0$ becomes

$$S_t = R_x. \tag{5.6}$$

Decomposing $R$ as follows,

$$R = U + W \tag{5.7}$$

where $g * U * g_*^{-1} \in \mathfrak{m}$ and $g * W * g_*^{-1} \in \mathfrak{k}$, we find $[S, R]_* = [S, U]_*$ and thus $[S, S_y]_* = [S, [S, U]_*]_*$. Using $(\mathrm{ad}A)^2 = -I$, which implies $(\mathrm{ad}S)^2 = -I$, we obtain $[S, S_y]_* = -U$, so (5.6) can be written as follows:

$$S_t = -([S, S_y]_* - W)_x. \tag{5.8}$$

[7] These expressions do not involve the $*$-inverse of $q$ and also apply to solutions for which $q$ is not $*$-invertible.

Furthermore, (5.7) becomes

$$g_*^{-1} * g_y = -[S, S_y]_* + W. \tag{5.9}$$

Equation (5.4) leads to $S_x = [S, g_*^{-1} * g_x]_*$ (since $A$ is constant), which implies

$$g_*^{-1} * g_x = -[S, S_x]_* \tag{5.10}$$

by use of $(\mathrm{ad}S)^2 = -I$. Together with the identity

$$(g_*^{-1} * g_x)_y - (g_*^{-1} * g_y)_x = [g_*^{-1} * g_x, g_*^{-1} * g_y]_* \tag{5.11}$$

the last two equations lead to

$$W_x = [[S, S_x]_*, W]_* + 2\,[S_x, S_y]_* - [[S, S_x]_*, [S, S_y]_*]_*. \tag{5.12}$$

The second term on the rhs can be rewritten as follows:

$$\begin{aligned}[][[S, S_x]_*, [S, S_y]_*]_* &= -[S, [[S, S_x]_*]_*, S_y]_* + [S, [[S, S_x]_*, S_y]_*]_* \\ &= [S_x, S_y]_* + [S, [[S, S_x]_*, S_y]_*]_* \end{aligned} \tag{5.13}$$

using again $(\mathrm{ad}S)^2 = -I$. (5.2) implies $g_*^{-1} * g_x \in g_*^{-1} * \mathfrak{m} * g$ and thus $[S, S_x]_* \in g_*^{-1} * \mathfrak{m} * g$. It follows that $[[S, S_x]_*, S_y]_* \in g_*^{-1} * \mathfrak{k} * g$. Since $S$ commutes with all elements of $g_*^{-1} * \mathfrak{k} * g$, we have $[S, [[S, S_x]_*, S_y]_*]_* = 0$. Hence

$$W_x = [S_x, S_y]_* + [[S, S_x]_*, W]_*. \tag{5.14}$$

Together with (5.8) this constitutes a (2 + 1)-dimensional matrix generalization of the Heisenberg ferromagnet equation, as we shall explain below.

**Example.** For $N = 2$ and $n = 1$ we have $S^2 = -\frac{1}{4}\,I$ and $W = u * S$ with a function $u$. Using

$$[[S, S_x]_*, u * S]_* = [[S, S_x]_*, u]_* * S + u * [[S, S_x]_*, S]_* = [[S, S_x]_*, u]_* * S + u * S_x \tag{5.15}$$

the system (5.8), (5.14) becomes

$$S_t = -([S, S_y]_* - u * S)_x \qquad u_x = -4\,[S_x, S_y]_* * S + [[S, S_x]_*, u]_*. \tag{5.16}$$

In the undeformed (commutative) case, the term $[[S, S_x]_*, u]_*$ disappears. Then we recover a system of equations which has been discussed in [7] (see also the references given there). Its nonlinear Schrödinger-type counterpart has been considered in [8]. Our equations (5.8), (5.14) thus constitute a matrix generalization of this system. Let us choose

$$g = \exp\left(-\frac{\mathrm{i}}{2}\sigma_2\,\varphi\right)\exp\left(\frac{\mathrm{i}}{2}\sigma_3\,t\right) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t/2}\cos(\varphi/2) & -\mathrm{e}^{-\mathrm{i}t/2}\sin(\varphi/2) \\ \mathrm{e}^{\mathrm{i}t/2}\sin(\varphi/2) & \mathrm{e}^{-\mathrm{i}t/2}\cos(\varphi/2) \end{pmatrix} \tag{5.17}$$

with a function $\varphi(x, y)$ and the Pauli matrices $\sigma_2$ and $\sigma_3$, we obtain

$$S = \frac{\mathrm{i}}{2}\begin{pmatrix} \cos\varphi & -\mathrm{e}^{-\mathrm{i}t}\sin\varphi \\ -\mathrm{e}^{\mathrm{i}t}\sin\varphi & -\cos\varphi \end{pmatrix}. \tag{5.18}$$

Then $[S_x, S_y] = 0$ and we can choose $W = 0$. Now (5.8) reduces to the sine–Gordon equation

$$\varphi_{xy} = \sin\varphi. \tag{5.19}$$

For $y = x$, (5.2) and the decomposition (5.7) imply $W = 0$. Without deformation, (5.8) then reduces to $S_t = -[S, S_{xx}]$, which is a matrix generalization [2] (see also [13]) of an equation which describes a one-dimensional continuous spin system (Heisenberg ferromagnet) [14]. Its equivalence with the nonlinear Schrödinger equation was demonstrated in [15] (see also [16]).

With $S = -\gamma_x$, (5.8) becomes[8]

$$\gamma_t = [\gamma_x, \gamma_{xy}]_* - W. \tag{5.20}$$

For $y = x$ we have $W = 0$. Without deformation, the last equation is then a generalization [2] of the Da Rios equation, which describes evolution of a thin vortex filament in a three-dimensional fluid [17] (see also [18, 19], in particular). Its equivalence with the NLS equation was first shown in [20].

## 6. Conclusions

Using bicomplex formalism [1], we obtained an extension of the Fordy–Kulish systems of matrix-NLS equations to three space-time dimensions. Moreover, corresponding equations on a noncommutative space-time are obtained by deformation quantization. We have shown that the resulting equations still possess an infinite set of conserved densities. Moreover, there is a gauge-equivalent generalized ferromagnet equation for all of these systems.

The Fordy–Kulish systems generalize the nonlinear Schrödinger equation, respectively the Heisenberg magnet or the Da Rios equation. The latter equations are associated with the simplest Hermitian symmetric space $SU(2)/S(U(1) \times U(1))$ in the series $SU(N)/S(U(n) \times U(N-n))$. The corresponding *extended* Fordy–Kulish system associated with this space reproduces the (2+1)-dimensional Myrzakulov system [7] and also the sine–Gordon equation, as shown in the previous section. There may be a way to understand the extended Fordy–Kulish systems as generalizations of the sine–Gordon equation in a similar way as they are quite obvious generalizations of the NLS equation. The extended Fordy–Kulish systems are matrix generalizations of the Myrzakulov system, of course.

In the case of matrix equations, the symmetric space structure of the Fordy–Kulish systems is just one way to sufficiently reduce the number of independent equations which result from the bicomplex conditions. There are different ways to avoid an overdetermined system. In [1] some other multi-dimensional systems of PDEs possessing a bicomplex formulation were presented.

Certain limits of string, D-brane and M theory generate field theories on noncommutative space-times which are obtained by a space-time deformation quantization (see [21] and the references given there). Among the various noncommutative models which arise in these and other ways, 'integrable' models will certainly be of special interest because of their highly distinguished properties. Although a suitable notion of integrability of a noncommutative field theory is not yet at hand, we think that the existence of an infinite set of conserved densities should be taken as a partial requirement. In the case of the deformations considered in this paper (see also [11, 12]), this requirement is satisfied. However, techniques to construct explicit solutions of the kinds of equation considered in this paper are still needed.

The relation between bi-Hamiltonian systems and bicomplexes has been clarified in [22]. The bicomplex structure is much more general, however, and neither presupposes a Hamiltonian structure nor implies one, as far as we can see. However, if a classical system admits a bi-Hamiltonian structure, it should be of interest to investigate its fate under a Moyal deformation which preserves the existence of an infinite set of conservation laws. We leave this for future work.

[8] An arbitrary matrix $V$ which does not depend on $x$ arises as a 'constant of integration'. It can be eliminated by a redefinition of $\gamma$.

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